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A THEORY OF THERMOVISCOPLASTICITY BASED ON INFINITESIMAL TOTAL STRAIN

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Abstract—A theory of three-dimensional infinitesimal isotropic thermoviscoplasticity is proposed and investigated in several homogeneous deformations. The theory is represented by both a mechanical constitutive equation and a constitutive assumption for the heat equation; these equations are separately postulated but are coupled through their common linear dependence upon the stress rate and strain rate tensors and the time rate of temperature change, and through their common nonlinear dependence upon the stress and strain tensors and the absolute temperature. Throughout this theory total strain is used and the strain tensor is not decomposed into a sum of elastic and inelastic contributions. The concept of the yield surface is not employed and the transition from initially linear thermoelastic behavior to nonlinear inelastic behavior is continuous.

Through qualitative discussion and quantitative examples, this proposed system of constitutive equations is shown to represent: initial linear thermoelastic behavior followed by inelastic rate sensitive work hardening; linear thermoelastic behavior between stress, strain and temperature in pure hydrostatic loading; initial cooling in uniaxial tension, initial heating in uniaxial compression, and initial isothermal behavior in torsion *prior* to inelastic deformation; subsequent self-heating in any monotonic loading following the onset of inelastic behavior; axial stress (strain) build-up due to temperature change uniformly induced by monotonic and cyclic torsional loading. Also represented are stress (strain) rate sensitivity; temperature rate sensitivity; creep and relaxation; and nonlinear dependence of the spacing between stress-strain curves at different stress (strain) rates upon the value of the stress (strain) rate tensor in constant-rate loading. Stress-strain curves corresponding to different constant strain (stress) rate loadings ultimately attain the same slope. Defined and physically meaningful limits are obtained in the cases of very slow and very fast loading rates in monotonic radial loading. Associated with limitingly slow loading is an equilibrium stress-strain curve, and in relaxation the stress relaxes to an equilibrium value associated with this curve.

Illustrative specializations to cases of uniaxial, torsional, pure hydrostatic and radial deformation histories are considered and numerical results are obtained by postulating specific choices of material functions, and by numerically integrating the resulting system of first-order nonautonomous, nonlinear stiff differential equations. Results are presented in terms of graphs to show the influence of various test histories and material parameters as well as the capability of the model.

In cyclic deformation the proposed equations of the model are modified according to previously established concepts to account for history dependence in the sense of plasticity. This is accomplished by updating a material parameter associated with the equilibrium stress-strain curves in such a manner that the material response predicted by the model remains continuous and smooth. The ability to represent hardening in materials is illustrated. In torsional cycling a significant net increase in temperature can be induced. A possible thermoinelastic explanation for the Ronay effect is given.

INTRODUCTION

Mechanical deformation of materials is accompanied by deformation-induced changes in temperature, which may or may not be equilibrated through heat conduction. Under special circumstances, however, such as rapid monotonic and cyclic loading, the temperature may rise considerably and may affect material deformation behavior. To model these phenomena a coupled mechanical and thermal theory of material behavior is necessary. In this paper we propose a coupled rate-dependent isotropic theory in the context of infinitesimal deformation based on total strain[1–3].

General theories of thermomechanical behavior of continua have been developed in[4–8]. A reduction to specific classes of solids, such as metals or polymers has not been attempted. Rate-independent thermoplasticity theories for the modelling of metal deformation behavior were proposed in[9–12]. Thermoviscoplasticity was considered in[13, 14]. The concept of a thermorheologically simple material has been proposed[15–19] which combines the effects of temperature and time into one parameter of process. Other models are based on thermoviscoelasticity[20–23].

Experimental information on deformation-induced temperature changes is not abundant[10–12, 24–29]. These experiments are difficult to perform and require sophisticated equipment which became available during the last decade. Such equipment was utilized in[30].

In developing the coupled theory of thermoviscoplasticity based on total infinitesimal strain we have considered the experimental results, our isothermal theory[1–3] and the first law of thermodynamics. This theory uses total strain only, regards rate dependence as fundamental, is nonlinear in the stress tensor, the strain tensor and the temperature, but is linear in the stress and strain rate tensors as well as in the temperature rate. The concept of a yield surface is not employed and the transition from the initial thermoelastic behavior to the subsequent thermoinelastic behavior is smooth.

The mechanical constitutive equation and the heat conduction are developed first. Then the properties of the proposed equations are studied qualitatively under adiabatic and spatially homogeneous conditions. Appropriate limits of the solutions of the nonlinear differential equations are obtained for limiting loading rates (extremely fast and extremely slow) and for large times. Next it is shown that instantaneous large changes in loading rate result in a continuous instantaneous thermoelastic response. The equations are then reduced to special, spatially homogeneous deformations, such as hydrostatic, uniaxial, biaxial and torsional states of stress. Numerical experiments using hypothetical but realistic material functions and constants show the predictive capability of the model. The case of torsional cycling is of special interest since it shows that torsional cycling introduces axial displacements or stresses depending on the boundary conditions due to deformation-induced temperature changes.

CONSTITUTIVE EQUATIONS

The mechanical constitutive equation

We propose this constitutive theory within the context of infinitesimal deformation, and we designate ϵ as the infinitesimal strain tensor and σ as the stress tensor. The absolute temperature is θ while the reference temperature corresponding to a state of equilibrium in the material is represented by θ_0. A superposed dot denotes differentiation with respect to time, and the usual comma-notation is employed to represent differentiation with respect to position coordinates. Following[1] we propose the isotropic thermomechanical constitutive equation

$$m[\sigma, \epsilon, \theta]\dot{\psi}_{ij} - k[\sigma, \epsilon, \theta]\dot{\sigma}_{ij} = \sigma_{ij} - G_{ij}[\epsilon, \theta] \tag{1}$$

with

$$\psi_{ij} = \frac{\nu\epsilon_{nn}}{(1+\nu)(1-2\nu)}\delta_{ij} + \frac{\epsilon_{ij}}{(1+\nu)} - \frac{\alpha}{(1-2\nu)}(\theta - \theta_0)\delta_{ij}. \tag{2}$$

In the above, square brackets denote "function of", α is the coefficient of thermal expansion which can be a function of temperature, and ν is Poisson's ratio which could be a function of the stress or strain tensors through their isotropic invariants and a function of temperature. These functions must be obtained from suitable experiments or may be approximated to be constant. For the isotropic theory the scalar functions $m[\,]$ and $k[\,]$ depend on the invariants of the stress and strain tensors. They are restricted to be positive, even, bounded and monotonic functions.

While other possibilities exist[1–3] we propose to make $m[\,]$ and $k[\,]$ dependent on the overstress magnitude

$$\Gamma = \{(\sigma_{ij} - G_{ij}[\,])(\sigma_{ij} - G_{ij}[\,])\}^{1/2} \tag{3}$$

and we propose to restrict the ratio†

$$\frac{m[\,]}{k[\,]} = E[\theta] \tag{4}$$

where E is the modulus of elasticity. These specializations are particularly useful for modelling metal deformation behavior[1–3].

Following[2] we call G the equilibrium stress-strain curves and require that G be odd in its first argument. We represent the function G by writing

$$G_{ij}[\epsilon, \theta] = \psi_{ij}\frac{g[\varphi, \theta]}{\varphi} \tag{5}$$

†Removing this restriction is discussed in the isothermal case[3].

Table 1. Representation of phi†

Invariant Subscript	φ
1	$\frac{(1.5e_{ij}e_{ij})^{1/2}}{(1+\nu)}$
2	$[\psi_{ij}\psi_{ij}]^{1/2}$
3	$\frac{(1.5e_{ij}e_{ij}\psi_{kl}\psi_{kl})^{1/4}}{(1+\nu)^{1/2}}$
4	$\left[\frac{\epsilon_{ij}\epsilon_{ij}}{(1+2\nu^2)}\right]^{1/2}$
5	$\left[\frac{1.5e_{ij}e_{ij}+\omega(\psi_{ij})^2}{(1+\nu)^2+\omega}\right]^{1/2}; \omega \geq 0$

†$e_{ij} = \epsilon_{ij} - (1/3)\epsilon_{kk}\delta_{ij}$.

where the symbol φ denotes a suitable, isotropic invariant of ϵ and temperature taken from Table 1. This invariant is normalized to reduce in uniaxial deformation to the uniaxial mechanical strain and as an example of a suitable representation we have the usual "effective strain"

$$\varphi = \varphi_1 = \left(\frac{1.5\, e_{ij}e_{ij}}{(1+\nu)^2}\right)^{1/2} \tag{6}$$

where $\mathbf{e}$ is the deviatoric strain tensor. We note that φ may depend upon both ϵ_{ij} and $\alpha(\theta-\theta_0)\delta_{ij}$. The function g is required to have the appearance of a uniaxial stress-strain curve, hence

$$g[0, \theta_0] = 0 \tag{7}$$

and

$$\frac{g[\varphi, \theta]}{\varphi} \cong E[\theta] \quad \text{when} \quad \varphi \cong 0. \tag{8}$$

In the numerical examples we use a specific representation of $g[\varphi, \theta]$ proposed in[31].

Equation (1) can be transformed into an equivalent integral equation

$$\sigma_{ij} = G_{ij}[E, \theta] + \int_0^t \left\{\frac{m[\,]}{k[\,]}\frac{\mathrm{d}\psi_{ij}}{\mathrm{d}\epsilon_{kl}} - \frac{\mathrm{d}G_{ij}}{\mathrm{d}\epsilon_{kl}}\right\} \cdot \left\{\exp - \int_\tau^t \frac{\mathrm{d}s}{k[\Sigma[s], E[s], \theta[s]]}\right\}\dot{\epsilon}_{kl}\, \mathrm{d}\tau \tag{9}$$

where we have used $\boldsymbol{\sigma}[t=0] = \mathbf{0}$, $\boldsymbol{\epsilon}[t=0] = \mathbf{0}$ and $\theta[t=0] = \theta_0$, and where we employ total differentiation with respect to strain

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon_{kl}} = \frac{\partial}{\partial\epsilon_{kl}} + \frac{\mathrm{d}\theta}{\mathrm{d}\epsilon_{kl}}\frac{\partial}{\partial\theta}. \tag{10}$$

We note that for constant material properties

$$\frac{\mathrm{d}\psi_{ij}}{\mathrm{d}\epsilon_{kl}} = \frac{1}{2(1+\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu\delta_{ij}\delta_{kl}}{(1+\nu)(1-2\nu)} - \frac{\alpha\delta_{ij}}{(1-2\nu)}\frac{\mathrm{d}\theta}{\mathrm{d}\epsilon_{kl}}. \tag{11}$$

Thermomechanical coupling

It is well known that mechanical working causes self-heating of materials; in some cyclic load applications inelastic self-heating can result in a temperature high enough to melt the material[29]. A constitutive theory should be able to represent these features through coupling of the deformation with the equation of heat conduction.

The deformation of metals is further characterized by initial thermoelastic response prior to inelastic and rate-sensitive behavior. During the initial elastic response the deformation produces cooling in uniaxial tension and heating in uniaxial compression, while pure torsion results in no temperature change (isothermal) for isotropic materials. Monotonic deformation into the inelastic region causes only self-heating. Under cyclic loading the temperature fluctuates around a monotonically increasing mean temperature[10–12, 24, 25, 29, 30].

The first law of thermodynamics for infinitesimal deformation is

$$\rho\dot{e} = \sigma_{ij}\dot{\epsilon}_{ij} - q_{i,i} + \rho R \tag{12}$$

where ρ is the constant mass density, e is the internal energy, $\mathbf{q}$ is the outward heat flux vector, $q_{i,i}$ is the divergence of the heat flux and R is an externally provided internal heat supply per unit mass.

In the absence of external heating ($R = 0$) and under adiabatic conditions ($q_{i,i} = 0$) the change in internal energy is entirely due to the mechanical working, i.e.

$$\rho(e[t] - e[t_0]) = \int_{t_0}^{t} \sigma_{ij}\dot{\epsilon}_{ij}\,\mathrm{d}\tau. \tag{13}$$

Thus the work done by the stresses is "stored" in the material. For a loading starting and ending at zero stress, i.e. $\sigma_{ij}[t] = \sigma_{ij}[t_0] = 0$, two forms of storage are possible;† change in temperature and change in the microstructure of the material[28, 32, 33].

Experimental evidence suggests that the energy of microstructure change is a small fraction ($\ll 10\%$) of the mechanical working[28, 32, 33] and for the purposes of this paper we assume that the mechanical working is completely transformed into heat which permits us to calculate the temperature increases due to self-heating quite well.‡

To represent this behavior we postulate that the internal energy rate is determined by

$$\rho\dot{e} = \rho\hat{C}[\theta]\dot{\theta} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{kk}^2 + \alpha\theta\sigma_{kk}\right] \tag{14}$$

where $\hat{C} = C[\theta] + (1/\rho)3E\alpha^2\theta/(1-2\nu)$ and where $C[\theta]$ is the specific heat of the material which usually is a slowly varying function of temperature. When mechanical behavior is thermoelastic, (14) represents the thermoelastic internal energy rate.

This representation of the energy rate is an exact differential and this permits us to write

$$e = \hat{e}[\sigma(t), \theta(t); \theta_0] \tag{15}$$

where we admit the parametric dependence of energy upon the initial equilibrium temperature and where we set $\hat{e}[0, \theta_0; \theta_0] = 0$.

Combining the first law of thermodynamics (12) with the constitutive assumption (14) permits us to write our equation of heat conduction

$$\rho\hat{C}[\theta]\dot{\theta} = \sigma_{ij}\dot{\epsilon}_{ij} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{kk}^2 + \alpha\theta\sigma_{kk}\right] - q_{i,i} + \rho R. \tag{16}$$

It is seen from (14) that for adiabatic conditions and for $R = 0$ the mechanical work in a cycle starting and ending at zero stress is transformed into temperature change.

Equations (1) or (9) and (2), (16) represent our coupled constitutive equations of thermoviscoplasticity. These follow from observations of material behavior and from the law of conservation of energy. In the case of nonadiabatic deformation we must additionally postulate a choice for the heat flux vector, and we take Fourier's isotropic heat conduction law $q_i = -K\theta_{,i}$ where K is the thermal conductivity.

The second law of thermodynamics may be written as[34]

$$\rho\theta\dot{\eta} \geq \rho R - q_{i,i} + \frac{q_i\theta_{,i}}{\theta}. \tag{17}$$

†If the loading would start or end at nonzero stress then mechanical energy such as the elastic energy could be stored in the deformed body.

‡Transformation of mechanical working into microstructure change will be considered in a future paper.

Equation (17) does not determine the rate of entropy, it rather specifies a lower bound for the entropy production rate. It is generally understood that this entropy inequality must not be violated. We select

$$\rho\theta\dot{\eta}[t] = \rho\hat{C}[\theta]\dot{\theta} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(1+\nu)}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{jj}^2 + \alpha\theta\sigma_{jj}\right) - \sigma_{ij}\dot{\epsilon}_{ij} + \frac{q_i\theta_{,i}}{\theta} + \Lambda \tag{18}$$

where Λ is a non-negative quantity which we chose not to specify further. With this choice the second law is not violated. We note that (18) is not an exact differential in stress and temperature, and consequently even in the case where Λ equals zero and the deformation is adiabatic, eqn (18) does not guarantee that the entropy depends only on the present stress and temperature.

Further discussions of thermodynamic aspects are given in[43].

Properties of the proposed coupled thermomechanical constitutive equations under adiabatic conditions

Equations (1), (2), (16) and a suitable law for the heat flux vector constitute a system of nonlinear first-order differential equations which are linear in the rates of the stress and the strain tensors and the temperature. For their solution we must employ the equations of motion and we must specify the external heat supply and the boundary conditions. If we consider zero heat supply and spatially homogeneous conditions, the heat flux vanishes and the equations of equilibrium are satisfied. We then must either specify the stress or strain tensor as a function of time. The temporal variations of the strain tensor (if the stresses are prescribed) or the stress tensor (if the strains are prescribed) and the temperature are then obtained from (1), (2), (16) and suitable initial conditions.

For $q_{i,i} = 0$ and $R = 0$ (16) is homogeneous of degree one in the rates and can be expressed as

$$\rho\hat{C}\frac{\mathrm{d}\theta}{\mathrm{d}\epsilon_{kl}} = \sigma_{kl} - \frac{\mathrm{d}}{\mathrm{d}\epsilon_{kl}}\left\{\frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{ii}^2 + \alpha\theta\sigma_{ii}\right\} \tag{19}$$

which for $\boldsymbol{\sigma} = \mathbf{0}$ reduces to

$$\frac{\mathrm{d}\theta}{\mathrm{d}\epsilon_{kl}} = -\frac{\alpha\theta}{\rho\hat{C}}\frac{\mathrm{d}\sigma_{mm}}{\mathrm{d}\epsilon_{kl}}. \tag{20}$$

For constant E, ν and α (19) can be rewritten as

$$\frac{\mathrm{d}\theta}{\mathrm{d}\epsilon_{kl}} = \frac{\sigma_{kl} - \dfrac{1+\nu}{E}\sigma_{ij}\dfrac{\mathrm{d}\sigma_{ij}}{\mathrm{d}\epsilon_{kl}} + \left(\dfrac{\nu}{E}\sigma_{mm} - \alpha\theta\right)\dfrac{\mathrm{d}\sigma_{ii}}{\mathrm{d}\epsilon_{kl}}}{\rho\hat{C} + \alpha\sigma_{nn}}. \tag{21}$$

From (20) we see that the model predicts initial cooling in tension, heating in compression and initial isothermal response in torsion prior to the onset of inelastic response.

Limiting behavior for extremely slow and fast loading

We impose a constant strain rate deformation and consider the limiting response predicted by the model in very fast and very slow loading at constant strain rate. Specifically we set

$$\dot{\epsilon}_{ij} = \beta Q_{ij} \tag{22}$$

with $\epsilon_{11} = \epsilon = \beta t$ and $Q_{11} = 1$, where β and Q are constants.

Substitution of (22) into the integral eqn (9) gives

$$\sigma_{ij}[\boldsymbol{\epsilon}, \theta] = G_{ij}[\boldsymbol{\epsilon}, \theta] + \int_0^{\epsilon}\left\{\frac{m[\,]}{k[\,]}\frac{\mathrm{d}\psi_{ij}}{\mathrm{d}\epsilon_{kl}} - \frac{\mathrm{d}G_{ij}}{\mathrm{d}\epsilon_{kl}}\right\}\left\{\exp -\frac{1}{\beta}\int_{\hat{\epsilon}}^{\epsilon}\frac{\mathrm{d}z}{k[\,]}\right\}Q_{kl}\,\mathrm{d}\hat{\epsilon}. \tag{23}$$

We examine the response under infinitely fast ($\beta \to \infty$) and infinitely slow ($\beta \to 0$) constant strain rate deformation by performing these limits using (23).

We obtain

$$\lim_{\beta \to 0} \sigma_{ij} = G_{ij}[\epsilon, \theta] \tag{24}$$

$$\lim_{\beta \to \infty} \sigma_{ij} = \int_0^{\epsilon} \frac{m[\,]}{k[\,]} \frac{d\psi_{ij}}{d\epsilon_{mn}} Q_{mn} \, d\hat{\epsilon}. \tag{25}$$

Using (4), eqn (25) simplifies to

$$\lim_{\beta \to \infty} \sigma_{ij} = \int_0^{\epsilon} E[\theta] \frac{d\psi_{ij}}{d\epsilon_{mn}} Q_{mn} \, d\hat{\epsilon} \tag{26}$$

and to

$$\lim_{\beta \to \infty} \sigma_{ij} = E\psi_{ij} \tag{27}$$

provided E is constant.

Substitution of any of (24)–(27) into (21) results in the appropriate representation of the heat equation corresponding to these limiting cases. From (24) we identify the function G as the equilibrium stress-strain curves of the material, as in the case of the isothermal theory[1]. We see that (4) does not guarantee a linear thermomechanical response for very fast deformation unless E is independent of temperature.

The limiting behavior obtained in (24) represents a lower bound to the mechanical response which the theory predicts at any finite strain rate, while the limits (25)–(27) represent an upper bound to this response. The same limits can be obtained for infinitely slow and infinitely fast constant stress rate loading. These limits indicate that as we deform the material at high rates the response tends more toward elastic behavior, while deforming the material at low rates results in a response closer to the equilibrium response manifested in G.

Limiting behavior at large times

We examine the behavior predicted by this theory at large times with constant stress rate or strain rate deformation. Through these considerations we obtain limiting bounds upon the response which the theory predicts, and while these limits may appear to imply large deformations, they are actually rapidly attained asymptotic limits. Previous investigations[1–3] and present numerical experiments indicate that these limits are attained by the response well within the strain magnitudes appropriate for infinitesimal deformation theory.

We define the difference between the stress and the equilibrium response G to be the overstress, and we determine the limit of the overstress as time goes to infinity in constant strain rate deformation. We use the integral equation to perform this limit and using (4) we obtain, following[1, 3]

$$\{\sigma_{ij} - G_{ij}\} = k[\Gamma, \theta]\left(E[\theta]\frac{d\psi_{ij}}{d\epsilon_{kl}} - \frac{dG_{ij}}{d\epsilon_{kl}}\right)\dot{\epsilon}_{kl} \tag{28}$$

where the braces denote the time limit of the overstress

$$\{\sigma_{ij} - G_{ij}\} \equiv \lim_{t \to \infty} (\sigma_{ij} - G_{ij}). \tag{29}$$

This limit is rapidly attained before the self-heating temperature increase is sufficient to influence the mechanical response noticeably.

Following a similar procedure we differentiate (9) in time obtaining the limit for the slopes

$$\lim_{t \to \infty} \frac{d\sigma_{ij}}{d\epsilon_{kl}} = \lim_{t \to \infty} \frac{dG_{ij}[\epsilon, \theta]}{d\epsilon_{kl}} \equiv A_{ijkl}. \tag{30}$$

We similarly consider the limiting overstress in constant stress rate deformation. We use the chain rule to replace the strain rate term in the integrand of the integral equation by writing

$$\dot{\epsilon}_{kl} = \frac{d\epsilon_{kl}}{d\sigma_{mn}}\dot{\sigma}_{mn}. \tag{31}$$

We proceed by taking the limit of the overstress, and we then substitute the "slope limit" (30) into this limiting overstress, noting that

$$\lim_{t\to\infty}\frac{d\epsilon_{kl}}{d\sigma_{mn}} = A^{-1}_{klmn}. \tag{32}$$

Using (32) we obtain the limiting overstress in constant stress rate deformation

$$\{\sigma_{ij} - G_{ij}\} = k[\Gamma, \theta]\left(E[\theta]\frac{d\psi_{ij}}{d\epsilon_{kl}} - \frac{dG_{ij}}{d\epsilon_{kl}}\right)A^{-1}_{klmn}\dot{\sigma}_{mn}. \tag{33}$$

In the particular case of our representation of the function G proposed in[31] we denote

$$E_s[\theta] = \lim_{\varphi\to\infty}\frac{g[\varphi, \theta]}{\varphi}. \tag{34}$$

Then we have, using (34)

$$A_{ijkl} = E_s[\theta]\frac{d\psi_{ij}}{d\epsilon_{kl}}. \tag{35}$$

and we obtain for (33)

$$\{\sigma_{ij} - G_{ij}\} = k[\Gamma, \theta](E[\theta] - E_s[\theta])\dot{\sigma}_{ij}/E_s[\theta]. \tag{36}$$

Because the function $k[\,]$ depends upon the special invariant defined in (3), eqns (28), (33) and (36) representing the limiting overstress are transcendental equations, indicating that the limiting overstress is *finite* and depends *nonlinearly* upon the constant strain rate or constant stress rate tensors. This property enables the modelling of arbitrarily close spacing of stress-strain curves obtained at widely different stress rates or strain rates, as shown in Fig. 1.† If the function k would be independent of Γ and θ the limiting overstress would depend linearly on the applied stress rates or strain rates[1, 3].

Behavior in the neighborhood of the stress and strain origin and of the initial temperature

The right-hand side of eqn (1) is initially zero in an undeformed material and we consequently obtain using (4) with constant material properties

$$\frac{d\sigma_{ij}}{d\epsilon_{kl}} = E\frac{d\psi_{ij}}{d\epsilon_{kl}}. \tag{37}$$

With this result and (20)

$$\frac{d\theta}{d\epsilon_{kl}} = -\frac{\alpha\theta E}{(1-2\nu)\rho C}\delta_{kl}. \tag{38}$$

This prediction corresponds to classical thermoelasticity.

Instantaneous large changes in strain rate or stress rate

Suppose that at any point $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$, θ the strain rate is instantaneously changed from $\dot{\epsilon}^-_{ij}$ to $\dot{\epsilon}^+_{ij}$

†Numerical simulations use the data given in Table 2.

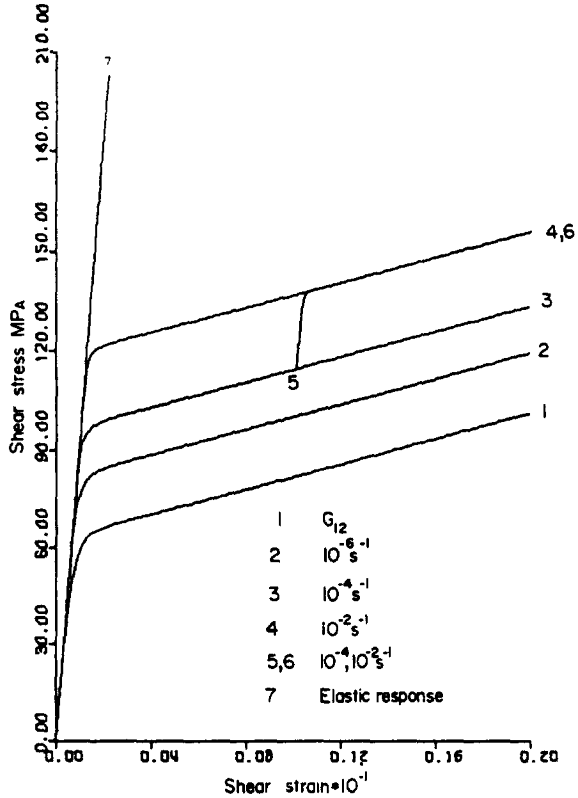

Fig. 1. Shear response under constant shear strain rates. The equilibrium stress–strain curve G_{12} and the elastic response are also shown together with a response to an instantaneous change in strain rate. The merging of curves 4 and 5, 6 indicates absence of strain rate history effect on stress.

where $\dot\epsilon_{ij}^-$ denotes the value of the strain-rate tensor prior to the jump change in strain rate and $\dot\epsilon_{ij}^+$ denotes the value of the strain-rate tensor following the radial jump in strain rate. We are interested in the change in the slopes $(d\sigma_{ij}^-/d\epsilon_{kl})$ to $(d\sigma_{ij}^+/d\epsilon_{kl})$ and $(d\theta^-/d\epsilon_{kl})$ to $(d\theta^+/d\epsilon_{kl})$ at the instant the strain rate is changed. From (1) and the chain rule we obtain using (4) and assuming constant E, ν and α†

$$\left(E\frac{d\psi_{ij}^+}{d\epsilon_{kl}} - \frac{d\sigma_{ij}^+}{d\epsilon_{kl}}\right)\dot\epsilon_{kl}^+ = \left(E\frac{d\psi_{ij}^-}{d\epsilon_{kl}} - \frac{d\sigma_{ij}^-}{d\epsilon_{kl}}\right)\dot\epsilon_{kl}^-. \tag{39}$$

To represent the radial jump in strain rate we write $\dot\epsilon_{ij}^+ = a\dot\epsilon_{ij}^-$ where "a" is a constant, and obtain

$$\frac{d\sigma_{ij}^+}{d\epsilon_{kl}} = \frac{1}{a}\frac{d\sigma_{ij}^-}{d\epsilon_{kl}} + E\left(\frac{d\psi_{ij}^+}{d\epsilon_{kl}} - \frac{1}{a}\frac{d\psi_{ij}^-}{d\epsilon_{kl}}\right). \tag{40}$$

Proceeding similarly, from (21) and (40) we obtain

$$\frac{d\theta^+}{d\epsilon_{kl}} = \frac{1}{a}\frac{d\theta^-}{d\epsilon_{kl}} - \frac{\left(1-\frac{1}{a}\right)\frac{E\alpha\theta}{1-2\nu}\delta_{kl}}{\rho\hat{C} - \frac{3\alpha^2 E\theta}{1-2\nu}} \tag{41}$$

If $|a| \gg 1$ (typical changes in strain rate are $10^2\, s^{-1}$ and higher) we have approximately

$$\frac{d\sigma_{ij}^+}{d\epsilon_{kl}} \cong E\frac{d\psi_{ij}^+}{d\epsilon_{kl}} \tag{42}$$

†Although it is lengthy to demonstrate mathematically the approximations also hold when the material properties are functions.

and

$$\frac{d\theta^+}{d\epsilon_{kl}} \cong -\frac{\alpha\theta E}{\rho C(1-2\nu)}\delta_{kl} \tag{43}$$

Comparing (42) with (37) and (43) with (38), respectively, we see that *large* radial positive or negative jumps in strain rate result in a thermoelastic slope of the stress-strain and the temperature-strain diagram; see Figs. 1 and 2.

To consider radial jump changes in stress rate we invert the constitutive equation (1) with (2) and using (4) in the case of constant E, ν and α we obtain

$$\dot{\epsilon}_{ij} = \frac{1}{Ek[]}((1+\nu)(\sigma_{ij}-G_{ij}) - \nu(\sigma_{kk}-G_{kk})\delta_{ij}) + \frac{1}{E}((1+\nu)\dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}) + \alpha\dot{\theta}\delta_{ij}. \tag{44}$$

We set $\dot{\sigma}^+_{ij} = b\dot{\sigma}^-_{ij}$ where "b" is a constant and obtain

$$\frac{d\epsilon^+_{mn}}{d\sigma_{kl}} = \frac{1}{b}\frac{d\epsilon^-_{mn}}{d\sigma_{kl}} + \alpha\delta_{mn}\left(\frac{d\theta^+}{d\sigma_{kl}} - \frac{1}{b}\frac{d\theta^-}{d\sigma_{kl}}\right) + D_{mnkl}\left(1-\frac{1}{b}\right) \tag{45}$$

where

$$D_{rskl} = \frac{1+\nu}{E}\left(\frac{1}{2}(\delta_{rk}\delta_{sl} + \delta_{rl}\delta_{sk}) - \frac{\nu}{1+\nu}\delta_{rs}\delta_{kl}\right). \tag{46}$$

Equations (16) and (45) yield

$$\frac{d\theta^+}{d\sigma_{kl}} = \frac{1}{b}\frac{d\theta^-}{d\sigma_{kl}} - \left(1-\frac{1}{b}\right)\frac{\alpha\theta}{\rho C}\delta_{kl} \tag{47}$$

Then for $|b| \gg 1$

$$\frac{d\epsilon^+_{rs}}{d\sigma_{kl}} \approx D_{rskl} + \alpha\delta_{rs}\frac{d\theta^+}{d\sigma_{kl}} \tag{48}$$

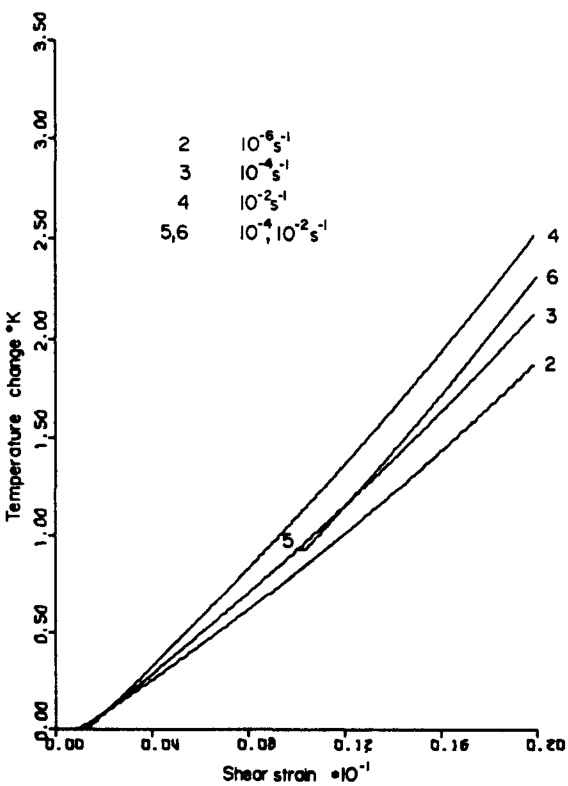

Fig. 2. Adiabatic temperature response for the constant strain rate tests 2–6 in Fig. 1. Curves 4 and 5, 6 never merge indicating presence of strain rate history effect on temperature.

and

$$\frac{d\theta^+}{d\sigma_{kl}} \approx -\frac{\alpha\theta}{\rho\hat{C}}\delta_{kl}. \tag{49}$$

The results in (48) and (49) are analogous to (42) and (43), respectively.

From this analysis we see that *large* changes $|b| \gg 1$ in the stress rate produce linear thermoelastic response. We note that this important property holds irrespective of the values of σ, ϵ and θ.

Reduction of the constitutive equations to special deformations under adiabatic conditions

In examining the predictions of this constitutive theory in special deformations we assume for simplicity and convenience that all material properties are constant unless otherwise specified. The predictions in the case of nonconstant material parameters follow directly from the theory but are cumbersome in their representation.

Pure hydrostatic deformation

In this case we have $\sigma_{ij} = \sigma\delta_{ij}$ and $\epsilon_{ij} = \epsilon\delta_{ij}$. Depending on the choice of the invariant φ in G different behaviors can be modelled. If a deviatoric φ from Table 1 is chosen then from (1) for constant material properties

$$\sigma_{ij} = E\psi_{ij} \tag{50}$$

and from (21)

$$\frac{d\theta}{d\epsilon_{kl}} = -\frac{\alpha\theta E}{(1-2\nu)\rho C}\delta_{kl}, \tag{51}$$

and we see that the behavior is linear thermoelastic. If we chose a non-deviatoric invariant then nonlinear, rate-dependent behavior will be obtained.

No information regarding actual material behavior under nonisothermal hydrostatic conditions is available and in keeping with tradition we choose a deviatoric invariant for metals. For other materials a nondeviatoric φ appears to be suitable.

Uniaxial deformation

In this case $\psi_{11} = x[t] = \epsilon - \alpha(\theta - \theta_0)$ and all other $\psi_{ij} = 0$; $\sigma_{11} = \sigma[t]$ and all other $\sigma_{ij} = 0$. Equations (1), (2), (6) and (16) reduce to

$$E\dot{x} - \dot{\sigma} = (\sigma - g[x, \theta])\frac{1}{k[\sigma - g, \theta]} \tag{52}$$

and

$$\frac{d\theta}{d\epsilon} = \frac{\sigma - \frac{d\sigma}{d\epsilon}\left(\frac{\sigma}{E} + \alpha\theta\right)}{\rho\hat{C} + \alpha\sigma} \tag{53}$$

or alternatively we write (53) as

$$(\rho\hat{C} - E\alpha^2\theta)\dot{\theta} = (\sigma + E\alpha\theta)(\sigma - g[x, \theta])\frac{1}{Ek[\,]} - E\alpha\theta\dot{\epsilon} \tag{54}$$

or

$$\rho\hat{C}\dot{\theta} = \frac{(\sigma - g[x, \theta])\sigma}{Ek[\,]} - \alpha\theta\dot{\sigma}. \tag{55}$$

The uniaxial stress response and temperature response under constant strain rate are shown in Figs. 3 and 4 (see Table 2 for data).

Torsion of a thin-walled tube

In pure torsion of a tube one usually specifies the shear stresses or shear strains and in this case we have $\psi_{12} = \psi_{21} = \gamma/(1+\nu)$ and $\sigma_{12} = \sigma_{21} = \tau$. Inelastic torsional deformation results in a temperature increase [10–12, 24–28], and this temperature change induces an axial "thermal

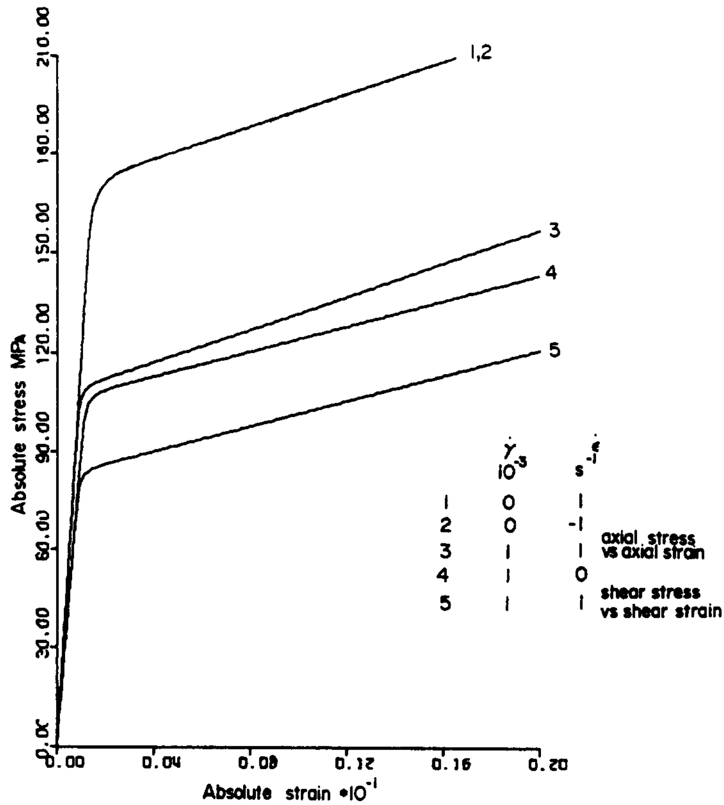

Fig. 3. Stress–strain response (absolute values) for tension, compression, shear and combined proportional loading with $\dot{\epsilon} = \dot{\gamma}$.

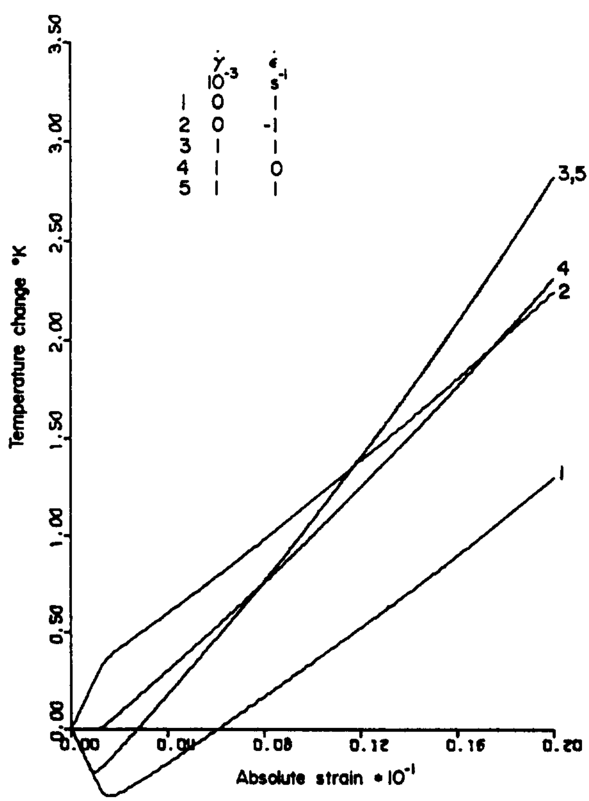

Fig. 4. Adiabatic temperature change in °K vs absolute strain for the tests depicted in Fig. 3.

Table 2. Material functions and properties used in the numerical experiments

1. General parameters

$\alpha = 14.4 \times 10^{-6}\,°\mathrm{K}^{-1}$ $\quad E = 120\,\mathrm{GPa}$

$\rho\hat{C} = 2.0\,\mathrm{MPa/°K}$ $\quad \nu = 0.3$

2. The $g[\,]$ function, see[31]

$$g[\varphi, \theta] = E_s\varphi + \frac{(E - E_s)}{2R\tanh[RX_f - 3]}\log_e\left[\frac{\cosh[U]}{\cosh[V]}\right]$$

where

$U = R(X_f + \varphi) - 3$ $\quad X_f = 0.003$†

$V = R(X_f - \varphi) - 3$ $\quad E_s = 2.5\,\mathrm{GPa}$

$\tanh(3) \cong 0.995$ $\quad \varphi = \varphi_1$ from Table 1

$R = R_{min} = 3.6/X_f$

In torsional cycling, Figs. 5–7, the parameter X_f is updated at successive occurrences of vanishing overstress. Successive values of X_f are: 0.0030, 0.0060, 0.0065, 0.0069, 0.00725, 0.00745, 0.00760, 0.00765.

3. The function $k[\Gamma]$ is obtained from[2] as

$$k = B \cdot \exp[21.275 \cdot \exp[-\Gamma/A]$$

$$A = 58.2818\,\mathrm{MPa}$$

$$B = 0.2296 \times 10^{-3}\,\mathrm{s}$$

†In cycling this parameter value is increased in the update-storage procedure.

strain". Consequently, in torsion we must also examine the axial component of the constitutive eqn (1) under specified boundary conditions. From (1)–(4) and (16) we have

$$\frac{E}{(1+\nu)}\dot{\gamma} - \dot{\tau} = \frac{1}{k[\Gamma, \theta]}(\tau - G_{12}), \tag{56}$$

$$E(\dot{\epsilon} - \alpha\dot{\theta}) - \dot{\sigma} = \frac{1}{k[\Gamma, \theta]}(\sigma - G_{11}) \tag{57}$$

and

$$(\rho\hat{C} - E\alpha^2\theta)\dot{\theta} = \frac{2(1+\nu)}{Ek[\,]}\tau(\tau - G_{12}) - E\alpha\theta\dot{\epsilon} + \frac{\sigma - G_{11}}{Ek[\,]}(\sigma + E\alpha\theta). \tag{58}$$

In the above (5) reduces to

$$G_{11} = (\epsilon - \alpha(\theta - \theta_0))\frac{g[\varphi, \theta]}{\varphi} \tag{59}$$

and

$$G_{12} = \frac{\gamma}{1+\nu}\frac{g[\varphi, \theta]}{\varphi}. \tag{60}$$

We have two choices for the axial boundary conditions:† zero stress or zero strain. If we require zero axial stress then σ and $\dot{\sigma}$ vanish for all time and this requires $\psi_{11} = 0$ and $G_{11} = 0$. Consequently under this boundary condition the axial deformation has the simple solution

$$\dot{\epsilon} = \alpha\dot{\theta}. \tag{61}$$

Equation (57) is then trivially satisfied and eqn (58) reduces to the torsional heat equation and

†Specification of nonzero axial stress or strain requires prior axial loading which we do not consider here.

contains only shear contributions

$$\rho\hat{C}\dot{\theta}=\frac{2(1+\nu)}{Ek[\,]}\tau(\tau-G_{12}). \tag{62}$$

Thus, when assuming this boundary condition the theory predicts a uniform elongation equal to $\alpha(\theta-\theta_0)$.

If we alternatively assume the zero strain boundary condition ($\epsilon=0, \dot{\epsilon}=0$) (57) reduces to

$$\dot{\sigma}+\frac{\sigma}{k[\,]}=-\alpha\left(E\dot{\theta}+(\theta-\theta_0)\frac{g[\varphi,\theta]}{k[\,]}\right) \tag{63}$$

while in (58) the $\dot{\epsilon}$-term vanishes. Equation (58) predicts a temperature increase due to the inelastic torsional deformation represented by (56). This temperature increase results in an axial compressive stress build-up according to (63) and this axial deformation can further contribute to the self-heating of the material as is clear by comparing (58) with (62).

In the first case the axial unit elongation is proportional to the temperature change (see eqn 61), which is generated through the torsional deformation through (62). However, in the second case there is zero axial unit elongation but there is a rate-dependent, inelastic and gradual compressive stress build-up predicted by (63).

Figures 1–4 show the stress-strain and temperature-strain response under *pure* torsional deformation of a thin-walled tube with boundary conditions of zero axial stress (eqns 56, 61 and 62 apply).

Strain-controlled torsional cycling

It is a fundamental assertion of our theory that viscoplastic behavior can be represented by discontinuously accounting for the deformation-induced microstructure change of the material[38, 39]. In our theory this accounting is accomplished at physically defined points. The values of stress, strain and temperature at these points are stored in material functions and simultaneously appropriate constant material parameters are updated. Storage without updating occurs in a cyclic steady state condition. Without this storage and updating our equations are of a nonlinear thermoviscoelastic nature. Through this procedure we represent thermoviscoplasticity with piecewise nonlinear thermoviscoelasticity.

Although storing and updating is discontinuous the response predicted by our theory remains continuous. We specify that storing and updating occur when any component of the overstress vanishes and subsequently changes sign. The rules for updating must be obtained from cyclic deformation experiments. Presently the rules for storing and updating are only developed for cyclic proportional loading.

To illustrate the procedure we simulate in Fig. 5 cyclic hardening in completely reversed strain controlled torsional cycling of an axially constrained thin-walled tube ($\epsilon=0$).

Equations (56), (58) (with $\dot{\epsilon}=0$) and (63) apply and the equilibrium functions are given by (59) and (60). At the first occurrence of vanishing torsional overstress the values of the equilibrium functions, strain and temperature are stored. Simultaneously the physically defined parameter X_f used in the representation of g (see [31]) is updated† to simulate hardening.‡ At the point of vanishing torsional overstress we denote the axial equilibrium function value, the shear equilibrium function value, the shear strain and the temperature as $G_{11}^{(1)}$, $G_{12}^{(1)}$, $\gamma^{(1)}$ and $\theta^{(1)}$. Departing the point of vanishing torsional overstress we have

$$G_{12}=\frac{\gamma-\gamma^{(1)}}{1+\nu}\frac{g^{(1)}[\varphi^{(1)},\theta]}{\varphi^{(1)}}+G_{12}^{(1)} \tag{64}$$

and

$$G_{11}=-\alpha(\theta-\theta^{(1)})\frac{g^{(1)}[\varphi^{(1)},\theta]}{\varphi^{(1)}}+G_{11}^{(1)} \tag{65}$$

†Additional possibilities exist through the updating of $k[\,]$.
‡If necessary more than one parameter can be changed (see [40]).

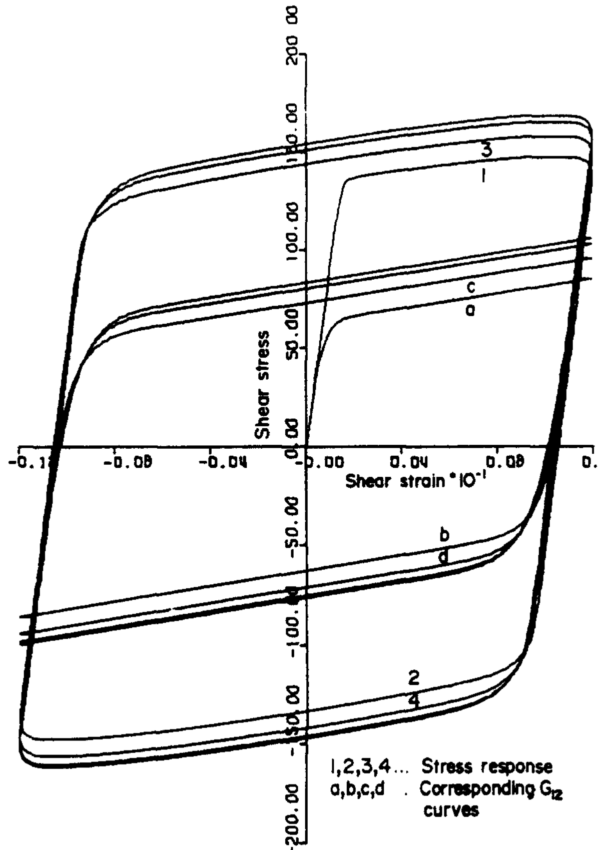

Fig. 5. Torsional strain cycling of a thin-walled tube under axial constraint ($\epsilon = 0$) at $\gamma = 0.012 \sin 2\pi t$. Cyclic hardening is simulated and corresponding equilibrium curves G_{12} are shown.

where $\varphi^{(1)}$ is determined using $\gamma - \gamma^{(1)}$ and $\theta - \theta^{(1)}$; $g^{(1)}$ represents the updated stress-strain curve.

At the subsequent occurrences of vanishing torsional overstress the storage-updating procedure is repeated.

The cyclic deformation shown in Fig. 5 results in the temperature increase depicted in Fig. 6 which in turn leads to the compressive stress build-up in Fig. 7.

DISCUSSION AND NUMERICAL EXPERIMENTS

The preceding constitutive theory represented by (1), (2), (16) with (3) reproduces in a unified way many of the qualitative features of nonisothermal, inelastic metal deformation generally attributed to thermoviscoplastic behavior. The capabilities of the theory include:

- Initial thermoelastic response at all loading rates prior to the onset of inelastic behavior.
- Subsequent deformation-induced temperature increase during the nonlinear (inelastic) mechanical response.
- Linear thermoelastic behavior at all values of stress and strain in the pure hydrostatic deformation field.
- Stress rate, strain rate and temperature rate sensitivity of the mechanical response.
- Creep and relaxation are treated in a unified way.
- Stress rate and strain rate sensitivity of the thermal response.
- Initial thermoelastic "slope" of both mechanical response and thermal response immediately upon the imposition of *large instantaneous* increases in the strain rate or stress rate.
- Defined mechanical and thermal behavior under limitingly slow and limitingly fast rates of deformation.
- Nonlinear dependence of steady-state mechanical response upon the applied constant stress rate or constant strain rate tensors (nonlinear spacing between stress-strain curves).

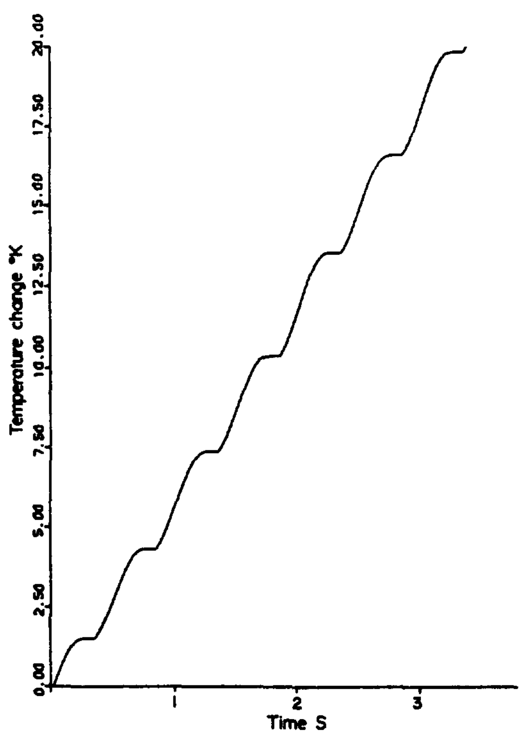

Fig. 6. Adiabatic temperature change corresponding to torsional cycling shown in Fig. 5.

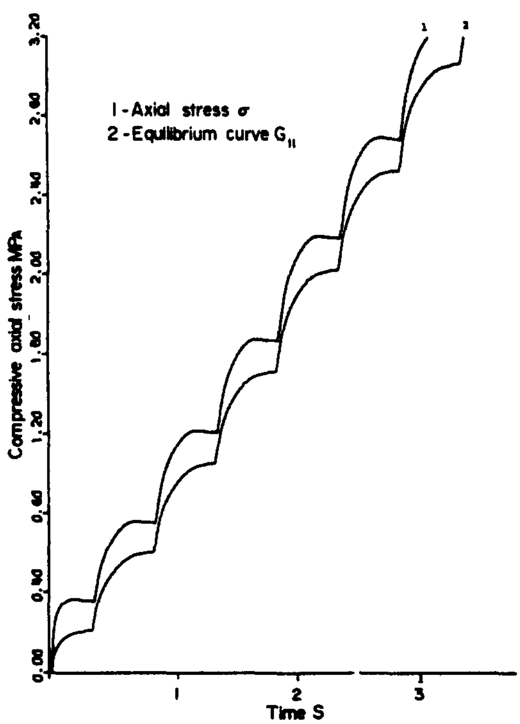

Fig. 7. Compressive axial stress increase in the thin-walled tube due to torsional cycling. The equilibrium curve G_{11} is also shown.

● Absence of strain rate history effect on stress; presence of strain rate history effect on temperature.

● Stress-strain curves obtained at different constant strain rates will ultimately have the same "slope".

● Temperature dependent "softening" of both initial elastic and subsequent inelastic response.

● Significant net increase of temperature under adiabatic cycling in torsion.

● Cycling hardening. Permanent set upon unloading to zero stress.

● In loading from and unloading to zero stress, energy of mechanical working is completely converted to temperature change.

● Temperature induced coupling of torsional and axial deformation as a possible source for the Ronay effect. Torsional cycling is demonstrated to result in temperature increase which isotropically induces either an axial strain field or an axial stress field.

In the above the terms "slope" and stress-strain refer to particular components of tensors specialized to one-dimensional homogeneous deformations such as the tensile or torsional tests.

The above capabilities are substantiated by the theory and are illustrated by numerical simulations. In the numerical examples we select material functions and integrate the resulting coupled nonlinear stiff differential equations using a computer program based on an algorithm developed in[41]. The material functions do not represent a specific real material. They are listed in Table 2 and are chosen to approximate the behavior of a low strength steel. Unless otherwise mentioned these functions are used in the computation of the responses shown in the following figures.

In Fig. 1 we simulate pure torsion at constant shear strain rates. The capability of nonlinear spacing in the model is clear, since increasing the strain rate two and four orders of magnitude in this simulation produces a small increase of steady-state flow stress. This behavior is predicted under the limits (28) and (33). Further, all stress-strain curves become parallel to each other and to the equilibrium function G_{12}, as predicted by the limits (30) and (35). At 1% shear strain an instantaneous jump in shear strain rate is imposed. The stress response immediately departs the point of the jump in rate with a thermoelastic slope, as predicted by eqns (42), (43). We also see in Fig. 1 and from the theory that there is no strain rate history effect on stress since the response corresponding to the jump increase rapidly merges from below with the response corresponding to uniform application of the high strain rate.

In Fig. 2 we have the temperature history corresponding to the shear deformations of Fig. 1. We see in Fig. 1 that deformation at increasing constant strain rates increases the region of apparent initial linear mechanical response. During the initial linear mechanical response the thermal response predicted by (16) or (21) is thermoelastic and consequently Fig. 2 shows isothermal response. Subsequently the inelastic torsional response results in self-heating and deformation at increasing strain rates results in increased heating. In Fig. 2 we see the temperature history corresponding to the jump in strain rate. While the stress response has no strain rate history effect, the thermal response has a clear strain rate history effect. This strain rate history effect on temperature response allows distinction between the continuous and jump histories.

Figure 3 indicates the predictions of this constitutive theory in tension, compression, torsion and proportional loading. In Fig. 4 we have the corresponding temperature responses predicted by (16). While the temperature and temperature rate are higher in compression than in tension (Fig. 4) the mechanical response in this case is negligibly affected and the two stress responses coincide on the graph in Fig. 3. We see that the initial response is thermoelastic but when the mechanical behavior becomes inelastic, self-heating rapidly occurs. In the case of proportional loading the mechanical response is decreased while the inelastic thermal response is increased relative to the pure shear and pure tensile response.

In Fig. 5 we simulate torsional cycling of a hypothetical hardening material using shear strain control; also shown is the torsional equilibrium function. When the torsional overstress vanishes we execute the updating procedure described previously. To simulate hardening the parameter X_f continues to change, see Table 2. In a cyclic steady state (closed loop) the value of X_f is unchanged although storage continues. We note that the response remains smooth even at the update points. If cycling would be stopped at zero stress, an aftereffect would occur. The strain magnitude would decrease until reaching a nontrivial equilibrium value determined by the intersection of the corresponding equilibrium curve with the strain axis. A permanent set is predicted by our model.

In Fig. 6 we see the adiabatic temperature response corresponding to the torsional cycling in

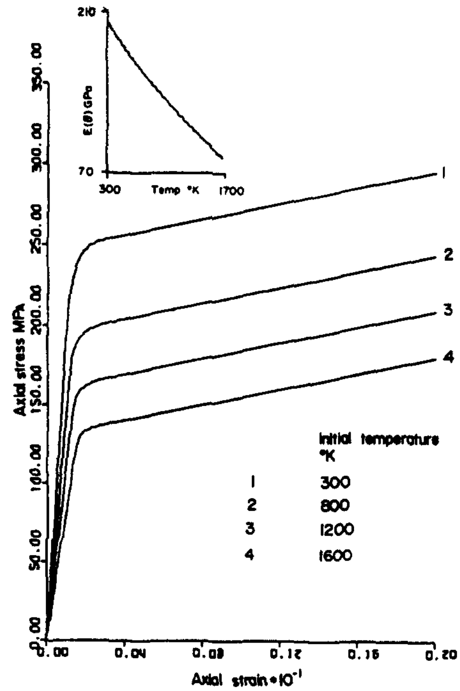

Fig. 8. The influence of initial temperature and temperature dependent elastic modulus on the stress–strain diagram in tension. $\dot{\epsilon} = 10^{-3}\,s^{-1}$. The temperature dependence of the elastic modulus is shown in the insert.

Fig. 5. We note the presence of both heating and isothermal phases during each torsional cycle similar to the experimental curves reported in[10–12, 24, 25].

This temperature response gives rise to a comprehensive stress build-up since the axial elongation was required to be zero. This axial stress build-up is shown in Fig. 7. While the magnitude of the axial stress is well within the linear region of Fig. 3, the behavior depicted in Fig. 7 is inelastic. In the case of zero axial stress an axial elongation occurs according to eqn (61). This thermally-induced behavior could be an explanation of the Ronay effect[35–37].

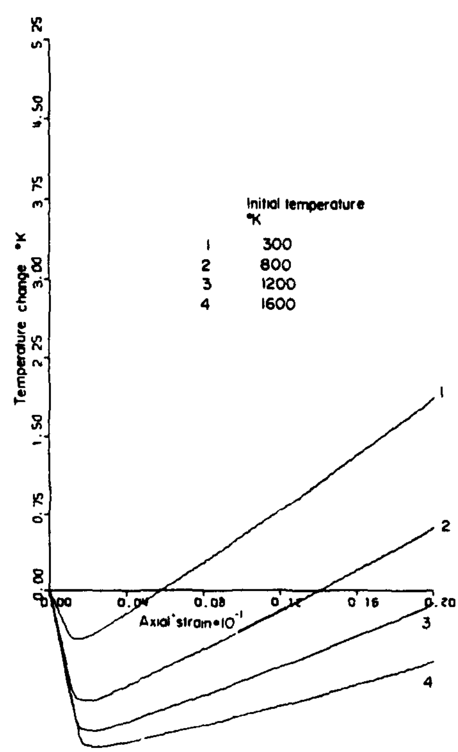

Fig. 9. The temperature change corresponding to the deformations shown in Fig. 8.

In Fig. 8 we have pure tensile response at one constant tensile strain rate ($10^{-3} s^{-1}$) but at different initial temperatures. The effect of "softening" the material arises through decrease of the elastic modulus $E[\theta]$ with increasing temperature depicted in the insert. In Fig. 9 we have the corresponding temperature response. Increasing initial temperature results in increasing intervals of elastic cooling prior to inelastic temperature increase as predicted by (16), (53) or (54).

Examination of Figs. 1–4, 8–9 shows that the temperature increase due to self-heating is usually small and of negligible influence on mechanical behavior. However, extensive adiabatic cycling may increase the temperature considerably with significant influence on mechanical behavior.

The equations proposed herein are linear in the temperature rate. In case of external heating the rate of heating will have an influence on the mechanical deformation. Such influences of temperature rate on mechanical deformation are reported in[42] for the case of external heating and mechanical deformation of a uniaxial bar. Equation (52) shows that both the rate of straining and the temperature rate have an effect on the stress. It predicts an increase in stress with increasing temperature rate as observed in[42]. In this case the heat source term R must be added to eqn (53).

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